On deep learning based approximation algorithms for partial differential equations

Arnulf Jentzen (ETH Zurich, Switzerland)

Joint works with

Weinan E (Beijing Institute of Big Data Research, China, Princeton University, USA, & Beijing University, China),

Máté Gerencér (IST Austria, Austria),

Jiequn Han (Princeton University, USA),

Martin Hairer (University of Warwick, UK),

Mario Hefter (University of Kaiserslautern, Germany),

Martin Hutzenthaler (University of Duisburg-Essen, Germany),

Thomas Kruse (University of Duisburg-Essen, Germany),

Thomas Müller-Gronbach (University of Passau, Germany),

Diyora Saliomva (ETH Zurich, Switzerland), and

Larisa Yaroslavtseva (University of Passau, Germany)

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Introduction

Consider

$$\frac{\partial u}{\partial t}(t,x) + f\big(x, u(t,x), (\nabla_x u)(t,x), (\operatorname{Hess}_x u)(t,x)\big) = 0 \qquad \text{(PDE)}$$

and $u(T,x) = g(x)$ for $t \in [0,T)$, $x \in \mathbb{R}^d$ where $T > 0$, $d \in \mathbb{N}$, $f\colon \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R}$, $g\colon \mathbb{R}^d \to \mathbb{R}$, $u \in C([0,T] \times \mathbb{R}^d, \mathbb{R})$ satisfies $u|_{[0,T)\times\mathbb{R}^d} \in C^{1,2}([0,T) \times \mathbb{R}^d, \mathbb{R})$. **Goal:** Solve (PDE) approximatively.

Applications: Pricing of financial derivatives, portfolio optimization, operations research

Approximations methods such as finite element methods, finite differences, sparse grids suffer under the curse of dimensionality.

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Linear PDEs

Theorem (Hairer, Hutzenthaler, & J 2015 AOP)

Let $T \in (0,\infty)$, $d \in \{4,5,\dots\}$, $\xi \in \mathbb{R}^d$. Then there exist globally bounded $\mu,\sigma \in C^\infty(\mathbb{R}^d,\mathbb{R}^d)$ such that for every probability space $(\Omega,\mathcal{F},\mathbb{P})$, every Brownian motion $W\colon [0,T]\times\Omega \to \mathbb{R}$, every solution $X\colon [0,T]\times\Omega\to\mathbb{R}^d$ of

$$\tfrac{\partial}{\partial t} X_t = \mu(X_t) + \sigma(X_t)\,\tfrac{\partial}{\partial t} W_t, \qquad t\in[0,T], \qquad X_0 = \xi,$$

every $Y^N\colon \{0,1,\dots,N\}\times\Omega\to\mathbb{R}^4$, $N\in\mathbb{N}$, with
$\forall\, N\in\mathbb{N}, n\in\{0,1,\dots,N-1\}\colon\ Y_0^N = X_0$ and

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(Euler-Maruyama approximations), and every $\alpha\in[0,\infty)$ we have

$$\lim_{N\to\infty}\left(N^\alpha\,\big\|\mathbb{E}[X_T] - \mathbb{E}[Y_N^N]\big\|\right) = \begin{cases} 0 & : \alpha = 0 \\ \infty & : \alpha > 0 \end{cases}.$$

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Let $T \in (0, \infty)$, $d \in \{2, 3, 4, \dots\}$, $\xi \in \mathbb{R}^d$, $(a_N)_{N \in \mathbb{N}} \subseteq \mathbb{R}$ *satisfy* $\lim_{N \to \infty} a_N = 0$. *Then there exist globally bounded* $\mu, \sigma \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$ *such that for every probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, *every Brownian motion* $W \colon [0, T] \times \Omega \to \mathbb{R}$, *every solution* $X \colon [0, T] \times \Omega \to \mathbb{R}^d$ *of*

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Theorem (Hefter & J 2017)

Let $T, \delta, \beta \in (0, \infty)$, $\gamma, \xi \in [0, \infty)$, *let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space, let* $W \colon [0, T] \times \Omega \to \mathbb{R}$ *be a Brownian motion, let* $X \colon [0, T] \times \Omega \to \mathbb{R}$ *be a solution of*

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Then there exists a $c \in (0, \infty)$ *such that for all* $N \in \mathbb{N}$ *we have*

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Nonlinear PDEs: Deep (2)BSDE method

Consider probability space $(\Omega, \mathcal{F}, \mathbb{P})$, Brownian motion $W\colon [0,T] \times \Omega \to \mathbb{R}^d$, normal filtration $\mathbb{F}$ generated by W, continuous and adapted $Y\colon [0,T] \times \Omega \to \mathbb{R}$ and $Z\colon [0,T] \times \Omega \to \mathbb{R}^d$ such that $\forall\, t \in [0,T]$ $\mathbb{P}$-a.s.:

$$Y_t = g(\xi + W_T) + \int_t^T f(Y_s, Z_s)\, ds - \int_t^T \langle Z_s, dW_s \rangle_{\mathbb{R}^d}. \qquad \text{(BSDE)}$$

Under suitable assumptions (Pardoux & Peng 1990 ...) it holds $\forall\, t \in [0,T]$ $\mathbb{P}$-a.s.:

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In particular, $\forall\, t_1, t_2 \in [0,T]$ with $t_1 \leq t_2$ it holds $\mathbb{P}$-a.s. that

$$Y_{t_2} = Y_{t_1} - \int_{t_1}^{t_2} f\big(Y_s, (\nabla_x u)(s, \xi + W_s)\big)\, ds + \int_{t_1}^{t_2} \langle (\nabla_x u)(s, \xi + W_s), dW_s \rangle_{\mathbb{R}^d}.$$

Consider $N \in \mathbb{N}$ and $0 = t_0 < t_1 < \ldots < t_N = T$ and observe that

$$Y_{t_{n+1}} \approx$$
$$Y_{t_n} - f\big(Y_{t_n}, (\nabla_x u)(t_n, \xi + W_{t_n})\big)\,(t_{n+1} - t_n) + \langle (\nabla_x u)(t_n, \xi + W_{t_n}), W_{t_{n+1}} - W_{t_n} \rangle_{\mathbb{R}^d}.$$

Consider probability space $(\Omega, \mathcal{F}, \mathbb{P})$, Brownian motion $W\colon [0,T]\times\Omega\to\mathbb{R}^d$, normal filtration $\mathbb{F}$ generated by W, continuous and adapted $Y\colon [0,T]\times\Omega\to\mathbb{R}$ and $Z\colon [0,T]\times\Omega\to\mathbb{R}^d$ such that $\forall\, t\in[0,T]$ $\mathbb{P}$-a.s.:

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$$Y_{t_2} = Y_{t_1} - \int_{t_1}^{t_2} f\big(Y_s, (\nabla_x u)(s, \xi + W_s)\big)\, ds + \int_{t_1}^{t_2} \langle (\nabla_x u)(s, \xi + W_s), dW_s \rangle_{\mathbb{R}^d}.$$

Consider $N \in \mathbb{N}$ and $0 = t_0 < t_1 < \ldots < t_N = T$ and observe that

$$Y_{t_{n+1}} \approx$$
$$Y_{t_n} - f\big(Y_{t_n}, (\nabla_x u)(t_n, \xi + W_{t_n})\big)\,(t_{n+1} - t_n) + \langle (\nabla_x u)(t_n, \xi + W_{t_n}), W_{t_{n+1}} - W_{t_n} \rangle_{\mathbb{R}^d}.$$

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Consider $\rho = 1 + 3Nd(d+1)$, let $\mathcal{R} \colon \mathbb{R}^d \to \mathbb{R}^d$ satisfy $\forall\, x = (x_1, \ldots, x_d)$:

$$\mathcal{R}(x) = (\max\{x_1, 0\}, \ldots, \max\{x_d, 0\}),$$

$\forall\, \theta = (\theta_1, \ldots, \theta_\rho) \in \mathbb{R}^\rho$, $v \in \mathbb{N}_0$, $k, l \in \mathbb{N}$ with $v + k(l+1) \leq \rho$ let $A^{\theta,v}_{k,l} \colon \mathbb{R}^l \to \mathbb{R}^k$ satisfy $\forall\, x = (x_1, \ldots, x_l)$:

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$\forall\, \theta \in \mathbb{R}^\rho$, $n \in \{0, 1, \ldots, N\}$ let $\mathcal{V}^\theta_n \colon \mathbb{R}^d \to \mathbb{R}^d$ satisfy

$$\mathcal{V}^\theta_n = A^{\theta,1+3nd(d+1)}_{d,d} \circ \mathcal{R} \circ A^{\theta,1+(3n+1)d(d+1)}_{d,d} \circ \mathcal{R} \circ A^{\theta,1+(3n+2)d(d+1)}_{d,d}$$

and $\forall\, \theta \in \mathbb{R}^\rho$ let $\mathcal{Y}^\theta \colon \{0, 1, \ldots, N\} \times \Omega \to \mathbb{R}$ satisfy $\mathcal{Y}^\theta_0 = \theta_1$ and

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We suggest to minimize

$$\mathbb{R}^\rho \ni \theta \mapsto \mathbb{E}\big[|\mathcal{Y}_N^\theta - g(\xi + W_T)|^2\big] \in [0, \infty). \qquad \text{(Optimization problem)}$$

Consider stochastic gradient descent-type approximations

$$\Theta = (\Theta^{(1)}, \ldots, \Theta^{(\rho)}) \colon \mathbb{N}_0 \times \Omega \to \mathbb{R}^\rho$$

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Consider $T, \gamma > 0$, $d, \rho, N \in \mathbb{N}$, $\xi \in \mathbb{R}^d$, $f\colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $g\colon \mathbb{R}^d \to \mathbb{R}$, $0 = t_0 < t_1 < \ldots < t_N = T$, probability space $(\Omega, \mathcal{F}, \mathbb{P})$, independent Brownian motions $W^m\colon [0,T] \times \Omega \to \mathbb{R}^d$, $m \in \mathbb{N}_0$, functions $\mathcal{V}_n^\theta\colon \mathbb{R}^d \to \mathbb{R}^d$, $\theta \in \mathbb{R}^\rho$, $0 \le n \le N$, for every $m \in \mathbb{N}_0$, $\theta = (\theta_1, \ldots, \theta_\rho) \in \mathbb{R}^\rho$ a function $\mathcal{Y}^{\theta,m}\colon \{0, 1, \ldots, N\} \times \Omega \to \mathbb{R}^k$ satisfying $\mathcal{Y}_0^{\theta,m} = \theta_1$ and $\forall\, n = 0, 1, \ldots, N-1\colon$

$$\mathcal{Y}_{n+1}^{\theta,m} = \mathcal{Y}_n^{\theta,m} - f\big(\mathcal{Y}_n^{\theta,m}, \mathcal{V}_n^\theta(\xi + W_{t_n}^m)\big)\,(t_{n+1} - t_n) + \big\langle \mathcal{V}_n^\theta(\xi + W_{t_n}^m), W_{t_{n+1}}^m - W_{t_n}^m \big\rangle_{\mathbb{R}^d},$$

for every $m \in \mathbb{N}_0$ a function $\phi^m\colon \mathbb{R}^\rho \times \Omega \to \mathbb{R}$ satisfying

$$\forall\, \theta \in \mathbb{R}^\rho\colon \qquad \phi^m(\theta) = \big|\mathcal{Y}_N^{\theta,m} - g(\xi + W_T^m)\big|^2,$$

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$$\forall\, \theta \in \mathbb{R}^\rho, \omega \in \Omega\colon \qquad \Phi^m(\theta, \omega) = (\nabla_\theta \phi^m)(\theta, \omega),$$

and $\Theta = (\Theta^{(1)}, \ldots, \Theta^{(\rho)})\colon \mathbb{N}_0 \times \Omega \to \mathbb{R}^\rho$ satisfying

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$$\forall\, \theta \in \mathbb{R}^\rho, \omega \in \Omega\colon \qquad \Phi^m(\theta, \omega) = (\nabla_\theta \phi^m)(\theta, \omega),$$

and $\Theta = (\Theta^{(1)}, \ldots, \Theta^{(\rho)})\colon \mathbb{N}_0 \times \Omega \to \mathbb{R}^\rho$ satisfying

$$\forall\, m \in \mathbb{N}\colon \qquad \Theta_m = \Theta_{m-1} - \gamma \cdot \Phi^m(\Theta_{m-1}).$$

We suggest for sufficiently large $N, \rho, m \in \mathbb{N}$ that $\Theta_m^{(1)} \approx u(0, \xi)$.

Numerical simulations

Implementations in PYTHON using TENSORFLOW
on a MACPOOK PRO 2.9 GHz (INTEL i5, 16 GB RAM)

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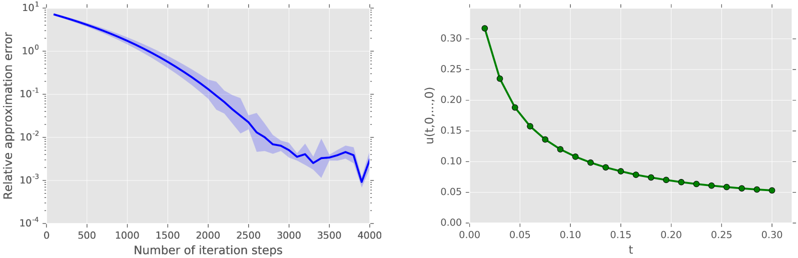
100-dimensional Allen-Cahn equation

Consider

$$\frac{\partial u}{\partial t}(t,x) = (\Delta_x u)(t,x) + u(t,x) - [u(t,x)]^3 \qquad \text{(Allen-Cahn)}$$

with $u(0,x) = \frac{1}{(2+0.4\|x\|^2)}$ for $t \in [0, \frac{3}{10}]$, $x \in \mathbb{R}^{100}$.



(a) Relative L^1-error for $u(\frac{3}{10}, 0) \approx 0.0528$ (b) Approximative plot of $u(t,0)$, $0 \leq t \leq \frac{3}{10}$

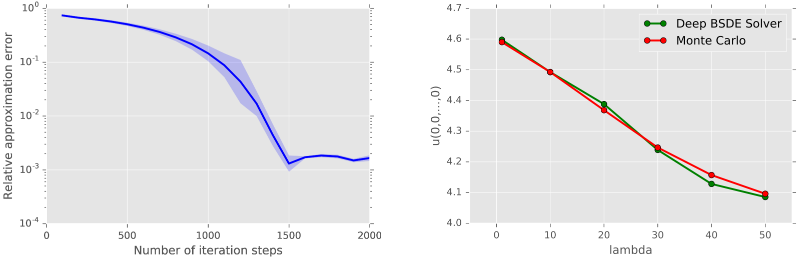
Deep BSDE method ($N = 20$, $\gamma = \frac{5}{10000}$): L^1**-error:** 0.3%, **Runtime:** 647 **seconds**.

100-dimensional Hamiltonian-Jacobi-Bellman equation

Consider

$$\frac{\partial u}{\partial t}(t,x) + (\Delta u_x)(t,x) = \lambda \|(\nabla_x u)(t,x)\|^2 \qquad \text{(HJB)}$$

with $u(1,x) = \frac{2}{(1+\|x\|^2)}$, $\lambda \geq 0$ for $t \in [0,1]$, $x \in \mathbb{R}^{100}$.



(a) Relative L^1-error when $\lambda = 1$

(b) Optimal cost against different λ

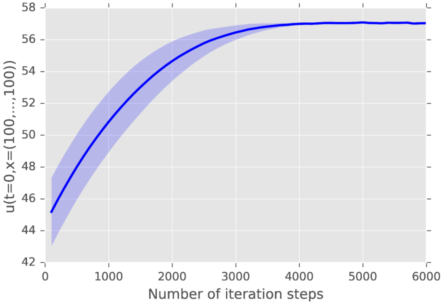
Deep BSDE method ($N = 20$, $\gamma = \frac{1}{100}$): L^1**-error:** 0.17%, **Runtime:** 330 **seconds**.

100-dimensional pricing model incorporating default risk

Duffie, Schroder, & Skiadas 1996, Bender, Schweizer, & Zhuo 2015:

$$\frac{\partial u}{\partial t}(t,x) + \bar{\mu}\,\langle x, (\nabla_x u)(t,x)\rangle_{\mathbb{R}^d} + \frac{\bar{\sigma}^2}{2}\sum_{i=1}^{d} |x_i|^2 \frac{\partial^2 u}{\partial x_i^2}(t,x) - Q(u(t,x))\,u(t,x) - R\,u(t,x) = 0$$

with $u(1,x) = \min_{1\leq i\leq 100} x_i$, $\bar{\mu} = R = 2\%$, $\bar{\sigma} = 20\%$ for $t \in [0,1]$, $x \in \mathbb{R}^{100}$.



Approximations for $u(0, 100, \ldots 100) \approx 57.3$ (defaultrisk excluded: ≈ 60.8)

Deep BSDE method ($N = 40$, $\gamma = \frac{8}{1000}$): L^1**-error:** 0.46%, **Runtime:** 617 **seconds**.

Fully-nonlinear PDEs:

- A 100-dimensional Black-Scholes-Barenblatt equation
- Nonlinear expectations of G-Brownian motions in 1 and 100 space-dimensions

All source codes available on GitHub **or** arXiv**:**

- E, Han, & J, *Deep learning-based numerical methods for high-dimensional parabolic partial differential equations and backward stochastic differential equations*. arXiv 2017. *Comm. Math. Stat. (2017)*
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Outlook: Other PDEs and Proofs!

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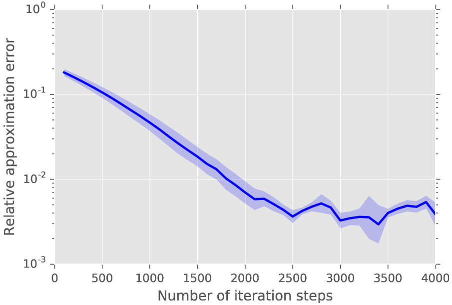
100-dimensional pricing model with different interest rates (Bergman 1995)

Consider $\bar{\sigma} = 20\%$, $R^l = 4\%$, $R^b = 6\%$ and for $t \in [0, 1/2]$, $x \in \mathbb{R}^{100}$:

$$\frac{\partial u}{\partial t} + \frac{\bar{\sigma}^2}{2} \sum_{i=1}^{d} |x_i|^2 \frac{\partial^2 u}{\partial x_i^2} - \min\left\{ R^b \left(u - \sum_{i=1}^{d} x_i \frac{\partial u}{\partial x_i} \right), R^l \left(u - \sum_{i=1}^{d} x_i \frac{\partial u}{\partial x_i} \right) \right\} = 0$$

with $u(1/2, x) = \max\{[\max_i x_i] - 120, 0\} - 2 \max\{[\max_i x_i] - 150, 0\}$.



Relative L^1-error for $u(0, 100, \ldots 100) \approx 21.299$

Deep BSDE method ($N = 20$, $\gamma = \frac{1}{200}$): L^1**-error:** 0.39%, **Runtime:** 566 **seconds**.

Pricing with default risk (Duffie, Schroder, & Skiadas 1996 AAP, Bender, Schweizer, & Zhuo 2015 MF)

Consider $\delta = \frac{2}{3}$, $\gamma^h = \frac{2}{10}$, $\gamma^l = \frac{2}{100}$, $v^h, v^l \in (0, \infty)$ satisfying $v^h < v^l$ and

$$Q(y) =$$
$$(1-\delta)\left[\gamma^h \mathbb{1}_{(-\infty, v^h)}(y) + \gamma^l \mathbb{1}_{[v^l, \infty)}(y) + \left[\frac{(\gamma^h - \gamma^l)}{(v^h - v^l)}\left(y - v^h\right) + \gamma^h\right] \mathbb{1}_{[v^h, v^l)}(y)\right].$$

- Bender et al. consider $v^h = 54$, $v^l = 90$ in the case $d = 5$.
- We consider $v^h = 50$, $v^l = 70$ in the case $d = 100$.

Plot of $\left\| \mathbb{E}\left[X_T\right] - \mathbb{E}\left[Y_N^N\right] \right\|$ for $T = 2$ and $N \in \{2^1, 2^2, \ldots, 2^{30}\}$.